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Scaling of overhangs appearing in fronts in higher dimensions

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Abstract. Directed percolation in higher dimensions serves as an example of how the distribution of overhangs gives information on the critical exponents of the system. Overhangs appear as jumps in the position of the front formed when a gradient in the control parameter is imposed along one of the spatial directions. By analyzing the overhang distribution in $(d+1)$-dimensional directed percolation, we determine the critical exponents β and $\nu_\perp$ for $d \leq 2$. In higher dimensions, the overhang distribution is insensitive to the critical region.

Suppose we have a system that undergoes a second-order phase transition when some control parameter is tuned. If we impose a spatial gradient in this control parameter, a front may occur separating the two phases. As shown by Sapoval *et al* [1] the front may contain information about the critical point, since its mean position will be in the area where the control parameter takes on its critical value.

Such gradients have been used to study diffusion fronts [1], percolation [2, 3], and critical points in directed processes such as cellular automata [4, 5]. In two dimensions, it was found that indeed the mean position of the front converged towards the critical value of the control parameter, and the width of the front showed scaling properties that could be related to a diverging correlation length. However, in three-dimensional percolation the situation turned out to be very different [6]. The reason for this is that the three-dimensional percolation actually possess *two* critical points rather than one, namely the percolation threshold for the 'present' sites whose connectivity is defined by the original lattice, and the percolation threshold for the 'absent' sites whose connectivity is defined on the *matching* lattice [7]. In two dimensions, these two critical points coincide. However, in three and more dimensions they do not. The position of the front therefore does not settle around the usual critical point, but, with the definition we use here, converges towards the critical point of the matching lattice, or to a position between the two critical points.

In [5] the concept of *overhangs* was introduced as an additional tool to study the front in connection with $(1+1)$-dimensional directed processes. It was shown that the distribution of these overhangs contains information on the critical behaviour of the order parameter, thus providing an effective way to determine the corresponding critical exponent β. This paper is a generalization of this work to higher-dimensional directed processes.

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In order to work with a specific example, we discuss *directed percolation* in $(d+1)$ dimensions [8]. In $(2+1)$ dimensions we imagine a cubic lattice oriented so that the [001]-direction points along the time direction. In higher dimensions, this geometry is generalized to a hypercubic lattice. Nodes in the lattice are given coordinates $(\boldsymbol{x}, t) = (x_1, x_2, \ldots, x_d, t)$. The linear size of the lattice is L along the spatial directions and T along the time direction. The nodes are occupied with probability p and empty with probability $1-p$. If there is a path between two nodes $(\boldsymbol{x}_1, t_1)$ and $(\boldsymbol{x}_2, t_2)$ where $t_2 > t_1$, that touches only occupied sites and does not move in the reverse time direction, then site 2 is connected to site 1. All sites connected to some common root belong to the same cluster. If all sites are occupied at $t = 0$, the order parameter is the density of sites belonging to the cluster that was initiated at $t = 0$. This cluster is called the 'infinite cluster'. In the thermodynamic limit L, $T \to \infty$, this density $P_\infty(p)$ will be zero for $p < p_c$ and finite for $p > p_c$. For p larger than, but close to p_c, the order parameter vanishes as

$$P_\infty(p) \sim (p - p_c)^\beta. \tag{1}$$

The critical behaviour is caused by the divergence of a spatial correlation length

$$\xi_\perp \sim |p - p_c|^{-\nu_\perp}. \tag{2}$$

Moving along a line in one of the spatial directions, we can record the distribution of 'holes' of length h occurring between sites belonging to the infinite cluster at p_c. This histogram, averaged over time, behaves as [9]

$$n_s(h) \sim h^{-(2-\beta/\nu_\perp)} \tag{3}$$

where $n_s(h)$ is to be interpreted as the probability *per site* of finding a hole of size h. As we have found no explicit derivation of (3) in the literature, and it is crucial for the rest of this paper, we present one here. The infinite cluster has a fractal dimension given by $d_f = d+1-\beta/\nu_\perp$. Thus, measured along a one-dimensional interval of length L, a mass M_1 scaling as $L^{d_f-(d-1)}$ will be found. Assuming that the infinite cluster consists of thin 'arms', the mass M_1 will be equal to the number of of holes along the interval

$$\left\langle \sum_i h_i{}^0 \right\rangle = M_1 \sim L^{d_f-(d-1)}. \tag{4}$$

Furthermore, we have that

$$\left\langle \sum_i h_i \right\rangle = L. \tag{5}$$

This equation states that the sum of the size of the holes must be equal to the length of the line, as the thickness of the fractal object itself is assumed to be negligible. Let us now make the assumption that the hole distribution measured along an interval L is of the form

$$n_s(h, L) \sim h^{-x} L^{y-1} \tag{6}$$

where x and y are two exponents to be determined. Combining (4) and (5) with (6) leads to

$$\left\langle \sum_i h_i{}^0 \right\rangle = L \int_1^L n_s(h, L)\, dh \sim L^y \int_1^L h^{-x}\, dh \sim L^{d_f-(d-1)} \tag{7}$$

and

$$\left\langle \sum_i h_i \right\rangle = L \int_1^L n_s(h, L) h\, dh \sim L^y \int_1^L h^{-x+1}\, dh \sim L. \tag{8}$$

Now, assuming that $(1 - x) < 0$ and $(2 - x) > 0$, we find

$$x = 2 + d_f - d = 2 - \beta/\nu_\perp \tag{9}$$

and

$$y = 2 + x = d_f - d = -\beta/\nu_\perp. \tag{10}$$

Thus, we find

$$n_s(h, L) \sim L \left(\frac{h}{L}\right)^{-(2-\beta/\nu_\perp)} \tag{11}$$

which is equivalent to (3).

We note that the assumption $(1 - x) < 0$ is equivalent to

$$1 - \beta/\nu_\perp > 0. \tag{12}$$

If this assumption is not fulfilled, it is not possible to find two exponents x and y that simultaneously fulfil (7) and (8), and we do not have a hole-size distribution $n_s(h)$ of the form (3). The reason for this breakdown when the inequality (12) is not fulfilled, is that the infinite cluster in this case is so tenuous that a one-dimensional line will only intersect the infinite cluster with a vanishing probability. This happens for $d \geq 2$ for directed percolation.

We impose a gradient along one of the spatial directions in the directed percolation problem, say the x_1 axis. The probability of finding an occupied site then varies as a function of position x_1 such that $p = 0$ for $x_1 = 0$ and $p = 1$ for $x_1 = L$. In general $p(x_1) = x_1/L$, i.e. $p(x_1)$ is proportional to the gradient $g = 1/L$. In order to measure the position of the front, we record the length of the string of empty sites starting at the leftmost site at $x = 0$ and ending at the first occurrence of an occupied site belonging to the infinite cluster. The reduced length of this string, denoted $x = x_1/L$, is a function of time and the $(d - 1)$-dimensional lattice perpendicular to the x_1-direction. The mean position of the front is then the average over time and the $(x_2, x_3, \ldots, x_d)$ coordinates.

In $(1+1)$ dimensions, the mean position of the front averaged in the time direction determines an effective percolation threshold, which converges to p_c with decreasing g as [1]

$$p_{\text{eff}}(g) = p_c + Ag^z + \cdots \tag{13}$$

where the exponent z is related to $\nu_\perp$. The width of the front, $w(g)$ behaves as [1]

$$w(g) \sim g^{-b} \tag{14}$$

where

$$b = \frac{\nu_\perp}{1 + \nu_\perp} \tag{15}$$

when the width is measured in units of $g = 1/L$.

We may also measure the mean position of the front in higher dimensions. Also in this case the mean position converges towards a well defined value p_m in the same way as in (13). However, p_m is *different* from p_c. For example, the directed site-percolation threshold in $(2+1)$ on the cubic lattice is [10] $p_c = 0.435\,25$, whereas we find $p_m = 0.317$. This surprising result may be explained along the lines of Rosso *et al* [6] for ordinary percolation in three dimensions, where the front settles around *the percolation threshold of the matching lattice* or *settles between the percolation* thresholds of the matching lattice and the original lattice. Chaté and Maneville [11] have measured the directed percolation threshold of the occupied sites on a bcc lattice, and find $p_c = 0.345$, while the percolation threshold of the empty sites on this lattice they place at $p_c = 0.315$. It is this threshold at which the mean position of the front, p_m, settles. They have, furthermore, measured the critical exponents associated with these two thresholds, and have found that they are equal to within numerical precision.

The width of the front always obeys a power law of the form (14). However, whether (15) is valid or not depends on whether the inequality (12) is fulfilled. If it is fulfilled, (15) applies. The reason for this may be found in the discussion following (12): if the inequality (12) is not fulfilled, the critical areas associated with the two percolation points of the occupied and empty sites are 'transparent', and the front will be found in the the non-critical area between them. If, however, the inequality is fulfilled, the front will settle at one of the critical points, and will reflect this in its scaling behaviour.

From numerical experiments, we find $b = 0.44(2)$ for $(2+1)$ dimensions (based on four samples of size $T = 25\,000$, $L = 10, 20, \ldots, 150$), $b = 0.47(2)$ for $(3+1)$ dimensions (based on five samples where $T = 25\,000$ and $L = 5, 10, \ldots, 45$), and $b = 0.48(2)$ in $(4+1)$ dimensions (based on four samples where $T = 10\,000$ and $L = 4, 6, \ldots, 22$). Using $\nu_\perp = 0.729(8)$ as determined by Grassberger [10] for $(2+1)$-dimensional directed percolation, (15) gives $b = 0.422(8)$. $(4+1)$ dimensions is upper critical dimension for directed percolation, and $\nu_\perp = 1/2$, [12] giving $b = 1/3$ in this dimension. Using Janssen's [12] ϵ expansion of the critical exponents in directed percolation, we find $\nu_\perp \approx 5/8$, giving $b \approx 0.38$ in $(3+1)$ dimensions. From these sources, the β exponent may be found: $\beta = 0.593(11)$ in $(2+1)$ dimensions [10], $\beta \approx 0.67$ in $(3+1)$ dimensions and $\beta = 1$ in $(4+1)$ dimensions [12]. Thus, using these data in (12), we see that the inequality is only fulfilled for $d \leq 2$. This is also the only case for which (15) works out numerically. In $(3+1)$ and $(4+1)$ dimensions, the exponent b turn out to be close to 1/2. This value may be argued as follows: in the area between the two critical points, the infinite cluster has a very convoluted structure, as this is an area where both 'present' and 'absent' sites percolate. It is therefore reasonable to assume that the structure of the front is dominated by *short-lived* fluctuations. This may be modelled by a simple Langevin equation [13],

$$\frac{\partial x_1}{\partial t} = p_m - g x_1 + \eta(x_2, \ldots, x_d, t) \tag{16}$$

where $x_1 = x_1(x_2, \ldots, x_d, t)$, and $\eta(x_2, \ldots, x_d, t)(= \eta(t))$ is a Gaussian noise term uncorrelated in time and in the orthogonal spatial directions $(x_2, \ldots, x_d)$, and normalized so that $\langle \eta(t) \rangle = 0$, and $\langle \eta(t_1)\eta(t_2) \rangle = \delta(t_1 - t_2)$. Equation (16) has the solution

$$x_1(x_2, \ldots, x_d, t) = \frac{p_{\mathrm{m}}}{g} + \mathrm{e}^{-gt} \int_0^t \eta(t')\mathrm{e}^{gt'} \mathrm{d}t' \tag{17}$$

so that $\langle x_1(t) \rangle = p_{\mathrm{m}}/g$. The width of the front behaves as

$$w^2(g) = \langle (x_1(x_2, \ldots, x_d, t) - x_1(x_2, \ldots, x_d, t))^2 \rangle = \frac{p_{\mathrm{m}}}{g} \left(1 - \mathrm{e}^{-2gt}\right). \tag{18}$$

Since we have assumed that the noise is uncorrelated in the orthogonal directions, averaging over these dimensions will not change the result. Thus comparing (18) with (14), shows that $b = 1/2$ when the above assumptions are valid.

Let us now turn to the overhang distribution. An *overhang*, $j(x_2, \ldots, x_d, t)$ is defined as

$$j(x_2, \ldots, x_d, t) = x_1(x_2, \ldots, x_d, t) - x_1(x_2, \ldots, x_d, t - 1). \tag{19}$$

Due to the geometry of the lattice the tips of the infinite cluster will move in steps of $j = \pm 1$ at each update. The front at position $(x_2, \ldots, x_d, t)$ is defined as the tip that has the smallest x_1 coordinate. The position of this front may move in steps larger than ± 1, when spatial or temporal foldings occur.

We calculate the time-averaged distribution of overhangs, $n(j, g, x_2, \ldots, x_d)$. Averaging over the orthogonal coordinates we obtain the distribution $n(j, g)$, which may be written in the scaling form [5]

$$n(j, g) = j^{-a} n_{\pm}(jg^b) \tag{20}$$

where $n_{\pm}(z)$ is a function that decays to zero faster than a power law for large arguments z, and approaches analytically a constant for small values of z. The scaling function $n_+(z)$ is to be used for *positive* values of j, $n_-(z)$ for negative values of j. The exponent b is the same as that of (14). This statement is based on the assumption that there is only one cut-off length determining the front. Furthermore, we conjecture that a is given by

$$a = 4 - d - \beta/\nu_{\perp} \tag{21}$$

as long as the inequality (12) is fulfilled. When (12) does not hold, our results suggest

$$a = 1 \tag{22}$$

and

$$b = 1/2. \tag{23}$$

Let us first assume that the inequality (12) is fulfilled. The position of the front is then determined by the percolation critical point of the empty sites on the matching lattice, and the overhangs are sampled in a critical region. Equation (18) is then based on the assumption that the overhang distribution *sample* the hole-size distribution of (3).

This is plausible since an overhang is essentially a 'hole' as defined above. In (3) the hole-size distribution is given as the probability per *site*. The hole-size distribution per *hole* becomes

$$n_{\rm h}(h) = \frac{n_{\rm s}(h)}{h} \sim h^{-(3-\beta/\nu_\perp)} \tag{24}$$

since the statistical weight of the different hole-sizes is not to depend on the size of the holes. Similarly, the overhang distribution $N(j, g, x_2, \ldots, x_d)$ is given as

$$n_\pm(j, g, x_2, \ldots, x_d) \sim j^{-(3-\beta/\nu_\perp)} n_\pm(jg^b). \tag{25}$$

Moreover, averaging this expression over the orthogonal coordinates will add an extra $(1-d)$-term to the exponent $3-\beta/\nu_\perp$ for the same reason that the exponent in (4) is different from that of (3): when averaging over the perpendicular coordinates, large overhangs will be given more weight than small ones, since these overhangs stretch out more in the orthogonal directions. These arguments lead to (21) for the exponent a.

On the other hand, when the inequality (12) is not fulfilled, the front is situated in the non-critical area between the at two critical points. We may then use the Langevin approach, (16), to determine the exponents a and b. The result $b = 1/2$ follows from the assumption that there is only one length scale in the front, and this length scale is proportional to the width of the front. It was demonstrated in (18) that the width of the front scales as $g^{-1/2}$. Even though the overhangs do not have a direct counterpart in the continuous-time description of the Langevin approach, we may approximate them by jumps in the position of the front at times $t-\epsilon$ and t. The second moment of these jumps behaves as

$$\langle [x_1(x_2, \ldots, x_d, t) - x_1(x_2, \ldots, x_d, t-\epsilon)]^2 \rangle = \frac{1}{g}\left(1 - {\rm e}^{-g\epsilon}\right). \tag{26}$$

Again using the assumption that the jumps are *uncorrelated* in the orthogonal dimensions $(x_2, \ldots, x_d)$, averaging over these coordinates does not change the result of (26). Comparing this with the second moment of the overhang distribution

$$\sum_{-\infty}^{+\infty} j^{2-a} n_\pm(jg^{1/2}) \sim g^{(3-a)/2} \tag{27}$$

gives (22). It should be noted that the above discussion—even though it is cast in the language of an approximative Langevin equation—is based on the assumption that when the front is in the area between the two critical regions, it behaves in such an erratic way that no spatial nor temporal long range correlations will be present. When the front is in the critical region, the opposite assumption applies, namely that the overhangs causing the fluctuations of the front are coherent structures which are spatially and temporarily extended.

We now turn to the numerical study of the overhang distribution. In [5], the scaling of (15), (20) and (21) was demonstrated numerically for $(1+1)$-dimensional directed percolation. From an analysis of the moments of the overhang distribution the two exponents were determined to be $a = 2.75(2)$ and $b = 0.53(1)$. These values agree very well with other estimates of the critical exponents. Using [14] $\nu_\perp = 1.0972(4)$ and [8] $\beta = 0.280(4)$ in (7) and (14), we find $a = 2.745(4)$, and $b = 0.5232(5)$.

We have repeated this analysis in $(2+1)$, $(3+1)$ and $(4+1)$ dimensions. The kth moment of the distribution of *positive* overhangs is defined as

$$\langle j^k(g)\rangle = \sum_{j=0}^{\infty} j^k\ \tilde{N}(j,g) = g^{b(a-k-1)} \sum_{j=0}^{\infty} j^k\ n_+(j) = A_k g^{-y(k)} \tag{28}$$

where we have used the scaling form given by (20). A_k is a constant and $y(k) = b(k+1-a)$. Since the zeroth moment should not diverge in the limit $g \to 0$, as this moment is nothing but the probability that the front moves in the direction of increasing p, we must have $a \geq 1$. Let us note that in $(1+1)$ dimensions, we have the additional constraint that the *first* moment must be non-divergent [5], leading to the inequality $a \geq 2$.

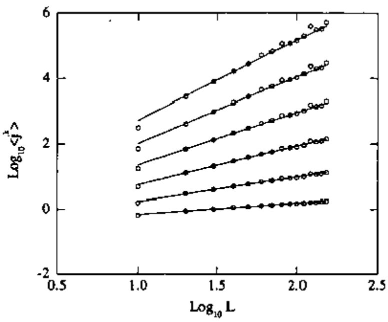

Figure 1. Figure 1 shows the moments $\log\langle j^k\rangle$ where $k = 1, 2, 3, 4, 5, 6$ *against* $\log L = -\log g$ for $(2+1)$-dimensional directed percolation. The slopes of these curves as found by the least-squares method are $y(1) = 0.33(1)$, $y(2) = 0.74(1)$, $y(3) = 1.18(2)$, $y(4) = 1.63(4)$, $y(5) = 2.09(5)$, and $y(6) = 2.55(7)$. The full lines are the predictions based on (15), (20) and (21) using the data of [10].

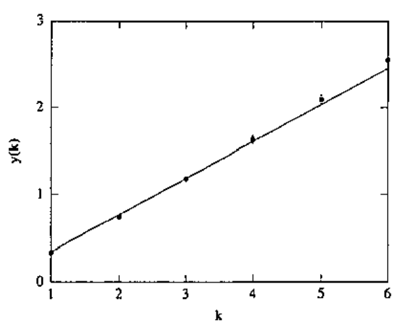

Figure 2. Figure 2 shows $y(k)$ as defined in text *against* k for $(2+1)$-dimensional directed percolation based on the data from figure 1. The full line is the form $y(k) = b(k-1=a)$ predicted by (15), (20) and (21) using a and b values from [10].

The first six moments, $k = 1,\ldots,6$ of the distribution of the positive overhangs are shown in figure 1 for $(2+1)$-dimensional directed percolation, along with the predictions from (15), (20) and (21) using the data of Grassberger [10]. In figures 2 and 3, we plot $y(k)$ *against* k for $(2+1)$, $(3+1)$ and $(4+1)$ dimensions. These plots

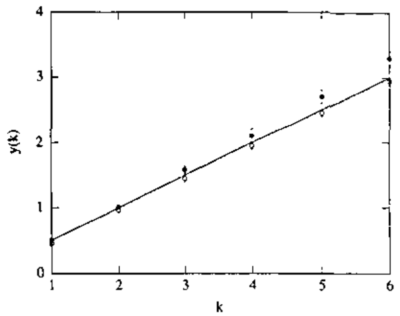

Figure 3. Figure 3 shows plots corresponding to figure 2 for $(3+1)$ (o) and $(4+1)$-dimensional (•) directed percolation. The full line has slope $1/2$ as predicted by (20), (22) and (23).

are based on the same runs as those referred to above in connection with the scaling of the width of the front. In these two figures we also show as full lines the predicted $y(k)$ from the above discussion. In $(2+1)$ dimensions, when using the data from [10], we predict $y(k) = 0.422(8)k - 0.0787(11)$, and in $(3+1)$ and $(4+1)$ dimensions, we predict $y(k) = k/2$.

We have in this paper demonstrated that the distribution of overhangs that occur when a gradient is introduced in a directed process possessing a critical point may yield valuable information on the nature of the critical point in question—also when the process occurs in higher dimensions, as long as the critical point is 'visible' to the front. This last situation is analogous to that of gradient percolation in higher dimensions [2, 6]. If the front 'sees' the critical point, the exponents $\nu_\perp$ and β may be deduced from the distribution.

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